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POLAR DECOMPOSITION IN RICKART C^* -ALGEBRAS

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Abstract

A new proof is obtained to the following fact: a Rickart C^* -algebra satisfies polar decomposition. Equivalently, matrix algebras over a Rickart C^* -algebra are also Rickart C^* -algebras.

Introduction.

In this paper we give new proof of the following result: all Rickart C^* -algebras satisfy polar decomposition. This fact was established in [2] by P. Ara and author by using a suitable factorization of the elements in the regular overring of a finite Rickart C^* -algebra.

New proof also uses the construction of the regular overring, but in a different way. In particular, we don't need the result of Goodearl, Lawrence and Handelman about algebras without one-dimensional representations.

The regular ring of measurable operators of a finite AW^* -algebra was constructed by S. K. Berberian in [3]. Later Saito modified Berberian's approach for general AW^* -algebras [11]. E. Christensen constructed and investigated a $*$ -algebra of measurable operators, associated to MSC C^* -algebras [5].

Handelman found a regular extension $Q(T)$ for a finite Rickart C^* -algebra T , using a technique of the module-homomorphisms on the essential countably generated ideals (instead of Berberian's coordinated sequences) [9]. It was established ([1], [10]) that a finite Rickart C^* -algebra T satisfies polar decomposition iff the bounded elements of $Q(t)$ belong to T . Goodearl, Handelman and Lawrence have proved that T satisfies polar decomposition in the case where T has no one-dimensional representations (see [8]).

P. Ara in [1], using his special construction, proved that left and right projections of element in a Rickart C^* -algebra are equivalent and the

polar decomposition problem in general Rickart C^* -algebras can be reduced to the finite case. In this work Ara also proved an equivalence of the following conditions for a Rickart C^* -algebra T :

- (i) T satisfies polar decomposition;
- (ii) The matrix algebras $M_n(T)$ over T are the Rickart C^* -algebras for all n ;
- (iii) The partial isometries of T are \aleph_0 -addable.

Finally, by development of the methods of [1], [8], it was proved in [2] that conditions (i)-(iii) are always fulfilled in Rickart C^* -algebras.

In [6], [7] was constructed a $*$ -algebra of measurable operators for a finite Rickart C^* -algebra and were proved some algebraic properties of this $*$ -algebra. We continue to develop this approach in order to solve the polar decomposition problem.

1. Preliminaries.

A $*$ -algebra A is *Rickart*, if for all $x \in A$ there exists a projection $e \in A$ such that $R(x) = \{a \in A \mid xa = 0\}$ is eA . Because of the involution, $L(x) = \{a \in A \mid ax = 0\} = Tf$ for some projection f . We shall write $e = RA(x)$, $f = LA(x)$, $1 - e = RP(x)$, $1 - f = LP(x)$ and $P(A)$ for the set of all projections of A .

The projections e and f are equivalent ($e \sim f$) in a $*$ -algebra A if $e = uu^*$, $f = u^*u$ for some partial isometry $u \in A$. A is finite if $p \sim 1$ implies $p = 1$. A Rickart C^* -algebra is a C^* -algebra that is also a Rickart $*$ -algebra. We recall some properties of the Rickart C^* -algebras.

Theorem 2.1. *A Rickart C^* -algebra satisfies the following properties:*

- (i) $P(T)$ is \aleph_0 -complete lattice partially ordered by $p \geq q$ iff $pq = q$ (see [4]).
If in addition T is finite then the lattice $P(T)$ is \aleph_0 -continuous [9, Cor. 1.1].
- (ii) $LP(x) \sim RP(x)$ for all $x \in T$ [1, Th. 2.5].
- (iii) For given sequences (e_n) and (f_n) of orthogonal projections such that $e_n \sim f_n$ for all $n \in \mathbf{N}$, we have $\bigvee_n e_n \sim \bigvee_n f_n$ [1].

The partial isometries are \aleph_0 -addable in a Rickart C^* -algebra T if for every sequence of partial isometries $\{w_n\}$ such that $\{w_n w_n^*\}$ and $\{w_n^* w_n\}$ are the sequences of orthogonal projections there exists a partial isometry w such that $w w_n^* w_n = w_n w_n^* w = w_n$.

2. Strongly dense domains.

Through this paper T denotes (if the opposite is not specified) a finite Rickart C^* -algebra.

A sequence of projections $(e_n) \subset P(T)$ is a strongly dense domain (SDD) in case $e_n \uparrow 1$. Let $e \in P(T)$, $x \in T$. We define $x^{-1}(e) = RA[(1 - e)x]$.

Proposition 2.1. *Let (e_n) and (f_n) are SDD, $x_n \in T$ such that $m \leq n$ implies $x_n e_m = x_m e_m$. Then a sequence $(t_n = x_n^{-1}(f_n) \wedge e_n)$ is a SDD.*

Proof: Let $d_n = x_n^{-1}(f_n)$. If $m \leq n$ then

$$(1 - e_n)x_n t_m = (1 - e_n)(1 - e_m)x_n e_m t_m = (1 - e_n)(1 - e_m)x_m t_m = 0,$$

so that $t_m \leq t_n$. Let $p \in P(T)$. We show that there exists a number k such that $t_k \wedge p \neq 0$. For that choose a number i such that $q = p \wedge e_i \neq 0$. If $x_i q = 0$, then $q \leq t_i$. Now let $x_i q \neq 0$. There exists $a \in T$ such that $h = x_i q a$ is non-zero projection [4, par. 8]. Observe that $x_i q = x_n q$ for all $n \geq i$ and $h' = f_k \wedge h \neq 0$ for sufficiently large k . For $k \geq i$ we have

$$(1 - f_k)x_k q a h' = (1 - f_k)x_i q a h' = (1 - f_k)h h' = 0.$$

Therefore $g = LP(q a h') \leq d_k$. In addition, $g \leq q \leq e_i \leq e_k$, hence $g \leq t_k$. Thus $p \wedge t_k \geq g \neq 0$. ■

Corollary 2.2. *If (e_n) and (f_n) are SDD, then $(e_n \wedge f_n)$ is also SDD.*

Proof: Put in Proposition 2.1 $x_n = 1$ for all n . ■

3. A ring of measurable operators.

An essentially measurable operator (EMO) is a pair of sequences (x_n, e_n) with $x_n \in T$, (e_n) an SDD, and such that $m \leq n$ implies $x_n e_m = x_m e_m$ and $x_n^* e_m = x_m^* e_m$. Two (EMO) (x_n, e_n) and (y_n, f_n) are equivalent, if there exists an SDD (g_n) such that $x_n g_n = y_n g_n$, $g_n x_n = g_n y_n$ for all $n \in \mathbf{N}$. Clearly that this relation is indeed equivalence relation (By Corollary 2.2). If (x_n, e_n) is (EMO), $[x_n, e_n]$ denotes its equivalence class. We call $[x_n, e_n]$ a measurable operator (MO) and denote by $S(T)$ the set of all (MO), and use the letters $\mathbf{x}, \mathbf{y}, \mathbf{z}, \dots$ for the elements of $S(T)$. Now we define the algebraic operations on $S(T)$. We put

$$\begin{aligned} [x_n, e_n] + [y_n, f_n] &= [x_n + y_n, e_n \wedge f_n] \\ \lambda[x_n, e_n] &= [\lambda x_n, e_n] \\ [x_n, e_n][y_n, f_n] &= [x_n y_n, e_n \wedge f_n], \\ [x_n, e_n]^* &= [x_n^*, e_n], \end{aligned}$$

where $k_n = f_n \wedge y_n^{-1}(e_n) \wedge e_n \wedge (x_n^*)^{-1}(f_n)$.

Summarizing,

Theorem 3.1. *The set $S(T)$ of all MO is a $*$ -algebra. The mapping $x \mapsto [x, 1] (x \in T)$ is a $*$ -isomorphism of T into Q , and $[1, 1]$ is a unity element for $S(T)$.*

We write $\bar{x} = [x, 1]$, for $x \in T$. The image of T in $S(T)$ is \bar{T} .

We recall the construction by Handelman of the $*$ -regular ring associated to a finite Rickart C^* -algebra. Let A be a unital ring. A right (left) ideal $E \subseteq A$ is *essential* if E has nontrivial intersection with any nonzero right (left) ideal of A . We say that E is *essential countably generated* (ecg) right ideal if there exist a sequence $\{t_n\}_{n \in \mathbf{N}} \subseteq A$ such that $\sum t_i A$ is essential in A . Similarly, we define left ecg ideal. It was proved in [9] that every ecg ideal of a finite Rickart C^* -algebra is generated by SDD.

Let T be a finite Rickart C^* -algebra. Consider the following pairs of mappings $[f, E; f_1, E_1]$, where f is right T -module homomorphisms from essential countably generated right ideal E , f_1 is left T -module homomorphism from essential countably generated left ideal E_1 , and they are balanced by the following condition: $e_1 f(e) = f_1(e_1)e$ for all $e \in E$ and all $e_1 \in E_1$. Two pairs $[f, E; f_1, E_1]$ and $[g, J; g_1, G_1]$ are equivalent if $f(x) = g(x)$ and $f_1(y) = g_1(y)$ for all $x \in E \cap J$ and all $y \in E_1 \cap J_1$. Let Q be the set of equivalence classes of just defined pairs. It was shown in [9] that Q is endowed with algebraic operations, and with respect to these operation Q becomes a $*$ -regular algebra.

Define mapping from $S(T)$ to Q . If $[x_n, e_n]$ is MO then $E = \bigcup_{n=1}^{\infty} e_n T$ ($E_1 = \bigcup_{n=1}^{\infty} T e_n$) is an essential countably generated right(left) ideal in T correspondently. Define a right T -module homomorphism $f : f(e_n t) = x_n e_n t$, where $e_n t \in E$. Obviously, $f(e_n t x) = f(e_n t)x$ for all $x \in T$. Let $e_n t = e_m s$ ($m \leq n$). Then

$$f(e_m s) = f(e_m)s = x_m e_m s = x_n e_m s = x_n e_n t = f(e_n t).$$

Thus this definition is correct. Similarly, we define a left T -module homomorphism $f_1 : E_1 \rightarrow T$, $f_1(t e_n) = t e_n x_n$. Now let $e \in E$, $e_1 \in E_1$, $e = e_m t$, $e_1 = t_1 e_n$. If $m \leq n$ then

$$\begin{aligned} e_1 f(e) &= t_1 e_n f(e_m t) = t_1 e_n x_m e_m t = t_1 e_n x_n e_m t \\ &= f_1(t_1 e_n) e_m t = f_1(e_1) e. \end{aligned}$$

By a similar argument $e_1 f(e) = f_1(e_1)e$. Therefore $[f, E, f_1, E_1] \in Q$. We shall denote just defined mapping by π . Then $\pi([x_n, e_n]) =$

$[f, E, f_1, E_1]$. Let $[x_n, e_n] = [x'_n, e'_n]$, $\pi([x'_n, e'_n]) = [f', E', f'_1, E'_1]$. Choose an SDD (p_n) such that $x_n p_n = x'_n p_n$, $p_n x_n = p_n x'_n$ for all $n \in \mathbf{N}$. Put $q_n = p_n \wedge e_n \wedge e'_n$. Note that $q_n \in E \cap E_1 \cap E' \cap E'_1$. We have $f(q_n) = x_n q_n = x_n p_n q_n = x'_n q_n = f(q_n)$. Thus $f = f'$ on $\bigcup_{n=1}^{\infty} q_n T$.

In the same way we obtain $f_1 = f'_1$ on $\bigcup_{n=1}^{\infty} T q_n$.

Theorem 3.2. *The mapping π is a $*$ -isomorphism from $S(T)$ onto Q .*

Proof: Let $[f, E, f_1, E_1] \in Q$, $E = \bigcup_{n=1}^{\infty} e_n T$, $E_1 = \bigcup_{n=1}^{\infty} T e_n$, (e_n) an SDD,

$$f(ex) = f(e)x, f_1(xe_1) = x f_1(e_1)$$

for all $e \in E$, $e_1 \in E_1$, $x \in T$. Put $f(e_n) = y_n$, $f_1(e_n) = z_n$. Obviously, $y_n e_n = y_n$, $e_n z_n = z_n$. Set

$$x_n = y_n + z_n - z_n e_n = y_n + z_n - e_n y_n$$

so that $x_n e_n = y_n$, $e_n x_n = z_n$ for all $n \in \mathbf{N}$. It is easy to see that $[x_n, e_n]$ is MO. Set $\pi([x_n, e_n]) = [g, E, g_1, E_1]$, where $g(e_n) = x_n e_n$, $g_1(e_n) = e_n x_n$. Then $g(e_n) = y_n = f(e_n)$, $g_1(e_n) = z_n = f_1(e_n)$, hence $[f, E, f_1, E_1] = [g, E, g_1, E_1]$. Thus π is surjective. Now we show that the mapping π preserves the algebraic operations. Let $[x_n, e_n]$, $[y_n, k_n] \in S(T)$. Put

$$\pi([x_n, e_n]) = [f, E, f_1, E_1], \pi([y_n, k_n]) = [g, J, g_1, J_1],$$

where

$$\begin{aligned} E &= \bigcup_{n=1}^{\infty} e_n T, E_1 = \bigcup_{n=1}^{\infty} T e_n, \\ J &= \bigcup_{n=1}^{\infty} k_n T, J_1 = \bigcup_{n=1}^{\infty} T k_n. \end{aligned}$$

We have (see [9, Section 2])

$$\begin{aligned} [f, E, f_1, E_1] + [g, J, g_1, J_1] &= [f + g, E \bigcap J, f_1 + g_1, E_1 \bigcap J_1], \\ [x_n, e_n] + [y_n, k_n] &= [x_n + y_n, e_n \bigwedge k_n]. \end{aligned}$$

Let $p_n = e_n \wedge k_n$ and $\pi([x_n + y_n, e_n \wedge k_n]) = [r, L, r_1, L_1]$. We can regard that

$$L = \bigcup_{n=1}^{\infty} p_n T, \quad L_1 = \bigcup_{n=1}^{\infty} T p_n,$$

$$r(p_n) = (x_n + y_n)p_n, \quad r_1(p_n) = p_n(x_n + y_n).$$

Since $(e_n \wedge k_n)T = (e_n T) \cap (k_n T)$ (see [9]) it follows $L = J \cap E$, $L_1 = J_1 \cap E_1$. In addition

$$r(p_n) = (x_n + y_n)p_n = (f + g)(p_n), \quad r_1(p_n) = p_n(x_n + y_n) = (f_1 + g_1)(p_n).$$

Consequently

$$[r, L, r_1, L_1] = [f + g, E \cap J, f_1 + g_1, E_1 \cap J_1].$$

Further, $[x_n, e_n][y_n, k_n] = [x_n y_n, t_n]$, where t_n is a suitable SDD. On the other hand,

$$[f, E, f_1, E_1][g, J, g_1, J_1] = [fg, g^{-1}E, g_1 f_1, f_1^{-1}J_1].$$

We shall establish that $\bigcup_{n=1}^{\infty} t_n T$ is an essential subideal in $g^{-1}E$. By the definition, $t_n = h_n \wedge g_n$, where

$$h_n = e_n \wedge (x_n^*)^{-1}(k_n), \quad g_n = k_n \wedge y_n^{-1}(e_n), \quad g^{-1}E = \{x \in J : g(x) \in E\}.$$

We have $t_n \leq g_n \leq k_n$, therefore $t_n \in J$ for all $n \in \mathbf{N}$. It remains to prove that $g(t_n) \in E$ for all n . Really,

$$\begin{aligned} g(t_n) &= g(g_n)t_n = g(g_n k_n y_n^{-1}(e_n))t_n = g(k_n)y_n^{-1}(e_n)g_n t_n \\ &= y_n k_n y_n^{-1}(e_n)g_n t_n = y_n y_n^{-1}(e_n)k_n g_n t_n \\ &= e_n y_n y_n^{-1}(e_n)k_n g_n t_n = e_n y_n y_n^{-1}(e_n)t_n, \end{aligned}$$

therefore $g(t_n) \in E$. So $t_n \in g^{-1}E$ for all n .

Similarly, we obtain $\bigcup_{n=1}^{\infty} T t_n \subset f^{-1}J_1$. Thus

$$[fg, g^{-1}E, f_1 g_1, f_1^{-1}J_1] = \left[fg, \bigcup_{n=1}^{\infty} t_n T, g_1 f_1, \bigcup_{n=1}^{\infty} T t_n \right].$$

It implies

$$\pi([x_n, e_n][y_n, k_n]) = \pi([x_n, e_n])\pi([y_n, k_n]).$$

Obviously,

$$\pi(\lambda[x_n, e_n]) = \lambda\pi([x_n, e_n]).$$

It is sthrightforward to check that $\pi([x_n, e_n]^*) = \pi([x_n, e_n])^*$. ■

Corollary 3.3. $S(T)$ is a Rickart $*$ -algebra and \aleph_0 -continuous ring.

Proof: It follows from Theorem 3.2 and [9, Th. 2.1]. ■

4. Some algebraic properties of $S(T)$. Cayley transform.

Lemma 4.1. If $\mathbf{x} = [x_n, e_n] \in Q$ and the x_n are all invertible then \mathbf{x} is invertible and $\mathbf{x}^{-1} = [x_n^{-1}, h_n]$ for a suitable SDD (h_n) .

Proof: Set $f_n = LP(x_n e_n)$. We show that (f_n) is a SDD. If $m \leq n$ then $f_n(x_m e_m) = f_n x_n e_m = f_n x_n e_n e_m = x_n e_n e_m = x_m e_m$, $f_m \leq f_n$. Since x_n is invertible then $RP(x_n e_n) = e_n$. We have $f_n \sim e_n$ [1, Th. 2.5]. As $p \sim q$ implies $1 - p \sim 1 - q$ for the projections p and q in a finite Rickart C^* -algebra, so $e_{n+1} - e_n \sim f_{n+1} - f_n$. Then by \aleph_0 -additivity,

$$1 = [\sup_n (e_{n+1} - e_n)] \bigvee e_1 \sim \left[\sup_n (f_{n+1} - f_n) \bigvee f_1 \right] = f.$$

Set $y_n = x_n^{-1}$. If $m \leq n$, then $y_n f_m = y_m f_m$. Really,

$$y_n x_m e_m = y_n x_n e_m = e_m = y_m x_m e_m,$$

hence $(y_n - y_m)x_m e_m = 0$ and $(y_n - y_m)f_m = 0$. Similary on setting $g_n = LP(x_n^* e_n)$, we have that (y_n) is a SDD and $y_n^* g_m = y_m^* g_m$ when $m \leq n$. Put $h_n = f_n \wedge g_n$, then $\mathbf{y} = [y_n, h_n]$ is MO, and $\mathbf{xy} = \mathbf{yx} = 1$. ■

Corollary 4.2. For any $\mathbf{x} \in S(T)$ an element $1 + \mathbf{x}^* \mathbf{x}$ is invertible.

Proof: It follows immediately from Lemma 4.1. ■

Lemma 4.3. If $\mathbf{x} = \mathbf{x}^*$, one can write $\mathbf{x} = [x_n, e_n]$ with $x_n^* = x_n$.

Proof: If $\mathbf{x} = [y_n, f_n]$, then $\mathbf{x} = 1/2(\mathbf{x}^* + \mathbf{x}) = [1/2(y_n^* + y_n), f_n]$. ■

Corollary 4.4. If $\mathbf{x} = \mathbf{x}^*$, then $\mathbf{x} + i$ is invertible.

Proof: Let $\mathbf{x} = [x_n, e_n]$, $x_n^* = x_n$; then $x + i = [x_n + i, e_n]$ and each $x_n + i$ is invertible. ■

Theorem 4.5. The formulas

$$\mathbf{u} = (\mathbf{x} - i)(\mathbf{x} + i)^{-1}, \quad \mathbf{x} = i(1 + \mathbf{u})(1 - \mathbf{u})^{-1}$$

define mutually inverse one-one correspondences between the self-adjoint elements $\mathbf{x} \in Q$, and the unitary elements \mathbf{u} for which $1 - \mathbf{u}$ is invertible.

Proof: It follows from Corollary 4.4. ■

We call this \mathbf{u} the Cayley transform of \mathbf{x} .

Lemma 4.6. *Let $\mathbf{x} = [x_n, e_n] \in S(T)$ and $x_n \longrightarrow x$ in norm, then $\mathbf{x} = \bar{\mathbf{x}}$.*

Proof: Evidently, $\|xe_n - x_n e_n\| = \|xe_n - x_k e_n\|$ for all $k \geq n$. Then $\|xe_n - x_n e_n\| \leq \|x - x_k\|$ for all $k \geq n$, $\|xe_n - x_n e_n\| = 0$, $xe_n = x_n e_n$. In just the same way, $e_n x = e_n x_n$. ■

Lemma 4.7. *Let $\mathbf{x} = [x_n, e_n] \in S(T)$. Then $\mathbf{x}e_n = \bar{\mathbf{x}}e_n$.*

Proof: Obvious. ■

5. The bounded measurable operators.

Let T be a finite Rickart C^* -algebra, $Q = S(T)$ denotes a $*$ -algebra of measurable operators of T .

An element $\mathbf{x} = [x_n, e_n] \in Q$ is bounded, if $\sup_n \|x_n\| \leq \infty$. Let B be a set of all bounded MO. It is clear that B is $*$ -algebra. Since $P(Q) \subset B$ hence B is Rickart $*$ -algebra. We define the mapping $\|\cdot\|_1 : B \ni \mathbf{x} \mapsto \|\mathbf{x}\|_1 = \inf \sup_n \{\|x_n\| \mid (x_n, e_n) \in \mathbf{x}\} \in \mathbf{R}$.

The bounded elements of $S(T)$ play a crucial role in the following discussion of the polar decomposition problem (or \aleph_0 -addability of the partial isometries, see Introduction) in a finite Rickart C^* -algebra. It is easy to see that the partial isometries of B are \aleph_0 -addable (Corollary 7.3). On the other hand, it is well known that the algebras B and \bar{T} coincide if T is AW^* -algebra [3]. We shall prove a similar result for a general Rickart C^* -algebra.

Theorem 5.1. *The mapping $\|\cdot\|_1$ is a C^* -norm.*

Proof: Let $\mathbf{x} = [x_n, e_n] \in B$. Clearly, $\|\mathbf{x}\|_1 \geq 0$. If $\|\mathbf{x}\|_1 = 0$ then for any $\varepsilon \geq 0$ there exists EMO $(x_n, e_n) \in \mathbf{x}$ such that $\sup_n \|x_n\| \leq \varepsilon$. Let $\mathbf{y} = [y_n, e_n] \in B$, $\sup_n \|y_n\| = \alpha$. We can choose $(x'_n, e'_n) \in \mathbf{x}$ with $\|x'_n\| \leq \varepsilon/\alpha$ for all $n \in \mathbf{N}$. Therefore $\|x_n y_n\| \leq \varepsilon$, $\|\mathbf{x}\mathbf{y}\|_1 = 0$.

Now assume that there exists $\mathbf{x} \in B$ such that $\mathbf{x} \neq 0$ and $\|\mathbf{x}\|_1 = 0$. Choose number n such that $\mathbf{x}e_n \neq 0$. By Lemma 4.7 $\mathbf{x}e_n = \bar{\mathbf{x}}e_n$. As it was shown above $\|\mathbf{x}e_n\|_1 = \|\bar{\mathbf{x}}e_n\|_1 = 0$. Let $a = x_n e_n$. By the definition of the norm $\|\cdot\|_1$ we have $\|a\|_1 = \inf \sup_n \{\|a_n\| \mid (a_n, k_n) \in \bar{a}\}$. For any $(a_n, k_n) \in \bar{a}$ there exists an SDD (p_n) such that $ap_n = a_n p_n$. Note $\|ap_n\| = \|a_n p_n\| \leq \|a_n\|$. Choose b such that $ba = e$ is a non-zero projection [4, par. 8]. Since $P(T)$ is \aleph_0 -continuous, there exists $k \in \mathbf{N}$ such that $q = e \wedge p_k \neq 0$.

Consequently

$$1 = \|q\| \leq \|ep_k\| = \|bap_k\| \leq \|ap_k\| \|b\|,$$

hence $\|ap_k\| \geq 1/\|b\|$. It follows $\|a_k\| \geq 1/\|b\|$, hence $\|\bar{a}\|_1 \neq 0$, a contradiction. Thus $\|\mathbf{x}\|_1 = 0$ implies $\mathbf{x} = 0$.

Obviously $\|\lambda\mathbf{x}\|_1 = \lambda\|\mathbf{x}\|_1$ for each $\mathbf{x} \in B$.

Further, let $\mathbf{x}, \mathbf{y} \in B$, $\mathbf{x} = [x_n, e_n]$, $\mathbf{y} = [y_n, f_n]$. Then

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\|_1 &= \inf \sup_n \{ \|c_n\| \mid (c_n, g_n) \in \mathbf{x} + \mathbf{y} \} \\ &\leq \inf \sup_n \{ \|x'_n + y'_n\| \mid (x'_n, e'_n) \in \mathbf{x}, (y'_n, f'_n) \in \mathbf{y} \} \\ &\leq \inf \sup_n \{ \|x'_n\| + \|y'_n\| \mid (x'_n, e'_n) \in \mathbf{x}, (y'_n, f'_n) \in \mathbf{y} \} \\ &= \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1. \end{aligned}$$

In just the same way, we get

$$\|\mathbf{xy}\|_1 \leq \|\mathbf{x}\|_1 \|\mathbf{y}\|_1.$$

From previous property we have $\|\mathbf{x}^*\mathbf{x}\|_1 \leq \|\mathbf{x}\|_1^2$. On the other hand, let $(b_n, q_n) \in \mathbf{x}^*\mathbf{x}$, $(s_n, k_n) \in \mathbf{x}$ for a suitable SDD k_n .

Hence there exists SDD (p_n) such that

$${}_np_n = s_n^*s_np_n, p_nb_n = p_ns_n^*s_n.$$

Let $t_n = p_n \wedge k_n \wedge q_n$. Then $(t_ns_n^*)(s_nt_n) = t_nb_nt_n$ and so $\|t_ns_n^*s_nt_n\| \leq \|b_n\|$. In addition, $[t_ns_n^*, f_n] = [s_n^*, k_n]$ for suitable SDD (f_n) , therefore $(t_ns_n^*, f_n) \in \mathbf{x}^*$. Hence for any EMO $(b_n, q_n) \in \mathbf{x}^*\mathbf{x}$ there exists EMO $(z_n, f_n) \in \mathbf{x}$ ($z_n = s_nt_n$) such that $\|z_n\|^2 \leq \|b_n\|$. Thus $\|\mathbf{x}\|_1^2 \leq \|\mathbf{x}^*\mathbf{x}\|_1$. ■

Corollary 5.2. *The norms $\|\cdot\|$ and $\|\cdot\|_1$ coincide on T .*

Proof: Let x be a positive element of T . By the definition, we have

$$\|\bar{x}\|_1 = \inf \sup_n \{ \|x_n\| \mid (x_n, e_n) \in \bar{x} \}.$$

Obviously $\|\bar{x}\|_1 \leq \|x\|$. Set $(x_n, e_n) \in \bar{x}$. Then there exists SDD p_n such that $x_np_n = xp_n$ for all n . Therefore $\|xp_n\| = \|x_np_n\| \leq \|x_n\|$. Choose a sequence of the positive numbers ε_n with $\varepsilon_n \uparrow \|x\|$. Set $\{x\}'' = C(K)$, for some Hausdorff space K . Put $U_n = \{a \in K : x(a) > \varepsilon_n\}$,

$$b_n(a) = \begin{cases} \frac{1}{x(a)}, & a \in \bar{U} \\ 0, & \text{otherwise.} \end{cases}$$

Since \overline{U}_n is clopen [4, par. 8] so $b_n(a) \in C(K)$ and $\|b_n(a)\| \leq \frac{1}{\varepsilon_n}$. As it was shown in Theorem 5.1, we can obtain that for each $n \in \mathbf{N}$ there exists a number $m(n)$ such that $\|x_m\| \geq 1/\|b_n\| \geq \varepsilon_n$ if $m \geq m(n)$. Therefore $\sup_m \|x_m\| \geq \varepsilon_n$ for all n . It follows that $\|\tilde{x}\|_1 \geq \varepsilon_n$ for all $n \in \mathbf{N}$. Therefore $\|\tilde{x}\|_1 \geq \|x\|$. Thus $\|\tilde{x}\|_1 = \|x\|$ for all positive $x \in T$. For arbitrary $x \in T$ we have $\|\tilde{x}\|_1^2 = \|\tilde{x}^* \tilde{x}\|_1 = \|x^* x\| = \|x\|^2$.

We shall use a notation \tilde{B} for a completion of B in the norm $\|\cdot\|_1$. In this connection $\tilde{\mathbf{x}}$ is an image of $\mathbf{x} \in B$ in \tilde{B} . ■

Lemma 5.3. *If $\mathbf{x} \in B$ and $\|\mathbf{x}\|_1 < 1$ then the series $\sum_{n \geq 0} \mathbf{x}^n$ converges to $(1 - \mathbf{x})^{-1} \in B$ in the norm $\|\cdot\|_1$.*

Proof: We can choose $(x_n, e_n) \in \mathbf{x}$ such that $\sup_n \|x_n\| < 1$. Then all the $1 - x_n$ are invertible. By Lemma 5.2 it follows that $1 - \mathbf{x}$ is invertible in Q and $(1 - \mathbf{x})^{-1} = [(1 - x_n)^{-1}, k_n]$ for suitable SDD (k_n) . Observe

$$\|(1 - x_n)^{-1}\| \leq \sum_{k \geq 0} \|x_n^k\| \leq \sum_{k \geq 0} \mu^k < \infty$$

for all n , where $\mu = \sup_n \|x_n\| < 1$. Thus $(1 - \mathbf{x})^{-1} \in B$. Identifying \mathbf{x} and $(1 - \mathbf{x})$ with their images $\tilde{\mathbf{x}}$ and $\widetilde{(1 - \mathbf{x})}^{-1}$ in C^* -algebra \tilde{B} , we get the statement of Lemma. ■

Lemma 5.4. *If $\mathbf{x} \in B$ then $\rho(\mathbf{x}) = \sup\{|\lambda| \mid \lambda \in \sigma(\mathbf{x})\} \leq \|\mathbf{x}\|_1$, where $\sigma(\mathbf{x})$ is a spectrum of \mathbf{x} .*

Proof: Let $|\lambda| > \|\mathbf{x}\|_1$, then applying Lemma 5.3 we obtain that the series $\lambda^{-1} \sum_{m \geq 0} (\mathbf{x} \lambda^{-1})^m$ converges to $(\lambda 1 - \mathbf{x})^{-1}$ in the norm $\|\cdot\|_1$ and lemma follows. ■

Lemma 5.5. *Let \mathbf{u} be a unitary element in B . Then*

$$\sigma(\mathbf{u}) \subset \{\lambda \in \mathbf{C} \mid |\lambda| = 1\}.$$

Proof: By Lemma 5.4 $\sigma(\mathbf{u}) \subseteq \{\lambda \in \mathbf{C} \mid |\lambda| \leq 1\}$. Since \mathbf{u} is invertible we have $\sigma(\mathbf{u}) = \overline{\sigma(\mathbf{u}^*)} = \overline{\sigma(\mathbf{u})}^{-1}$. It follows that $\sigma(\mathbf{u}) \subseteq \{\lambda \in \mathbf{C} \mid |\lambda| = 1\}$. ■

Lemma 5.6. *If $\mathbf{x} \in B$, $\mathbf{x} = \mathbf{x}^*$ then $\sigma(\mathbf{x}) \subseteq [-\|\mathbf{x}\|_1, \|\mathbf{x}\|_1]$.*

Proof: The proof is similar to the case of C^* -algebras. ■

6. Module of a self-adjoint element from B .

We call an element $x \in B$ positive, $\mathbf{x} \geq 0$, if $\tilde{\mathbf{x}} \geq 0$.

The goal of this section is to prove that for any self-adjoint element $\mathbf{x} \in B$ there exists an unique positive $\mathbf{y} \in B$ such that $\mathbf{y}^2 = \mathbf{x}^2$.

Theorem 6.1. *Let \mathbf{u} be a unitary element of B . Then the mapping $\overline{T} \ni \bar{x} \mapsto \mathbf{u}\bar{x}\mathbf{u}^* \in B$ is a $*$ -automorphism of a finite Rickart C^* -algebra \overline{T} .*

Proof: Set $A = \mathbf{u}\overline{T}\mathbf{u}^*$. Obviously A is a $*$ -algebra with a C^* -norm $\|\cdot\|_1$. Let $\{\mathbf{x}_n\}$ be a $\|\cdot\|_1$ -fundamental sequence in A . Then there exists a sequence $\{t_n\}$ such that $x_n = \mathbf{u}\bar{t}_n\mathbf{u}^*$. Since

$$\|t_n - t_m\| = \|\bar{t}_n - \bar{t}_m\|_1 = \|\mathbf{u}^*(\mathbf{u}\bar{t}_n\mathbf{u}^* - \mathbf{u}\bar{t}_m\mathbf{u}^*)\mathbf{u}\|_1 \leq \|\mathbf{x}_n - \mathbf{x}_m\|_1,$$

hence the sequence $\{\bar{t}_n\}$ is fundamental in T . Let $t = \lim_{n \rightarrow \infty} t_n$. Then clearly that the sequence $\{\mathbf{u}\bar{t}_n\mathbf{u}^*\}$ converges to $\mathbf{u}\bar{t}\mathbf{u}^*$ in the norm $\|\cdot\|_1$. Thus A is a C^* -algebra. Clearly, that $P(A) \subset P(Q)$. On the other hand, any projection $e \in P(T)$ can be written as $\mathbf{u}(\mathbf{u}^*e\mathbf{u})\mathbf{u}^*$. Since $\mathbf{u}^*e\mathbf{u} \in P(Q) = P(T)$ ([9]) we conclude that $P(A) = P(T)$. By spectral theory, it follows that $A = \overline{T}$. ■

The next Corollary is a key result in proving an existence of a module of self-adjoint element of B .

Corollary 6.2. *Let \mathbf{u} be a unitary element of B , $t \in T$. Then $\bar{t}\mathbf{u} \in \overline{T}$ implies $\mathbf{u}\bar{t} \in \overline{T}$.*

Proof: Since $\mathbf{u}\bar{t} = \mathbf{u}(\bar{t}\mathbf{u})\mathbf{u}^*$, by using Theorem 6.1 we have $\mathbf{u}\bar{t} \in \overline{T}$. ■

Proposition 6.3. *Let $\mathbf{x} \in B$ and SDD (e_n) such that $\mathbf{x}e_n, e_n\mathbf{x} \in \overline{T}$ for all $n \in \mathbf{N}$. Then $\mathbf{x} = [y_n, e_n]$, where $\bar{y}_n = \mathbf{x}e_n + e_n\mathbf{x} - e_n\mathbf{x}e_n$.*

Proof: Let $\mathbf{x} = [x_n, p_n]$, $q_n = p_n \wedge e_n$. By using Lemma 4.7,

$$\bar{x}_n q_n = \mathbf{x}q_n = (\mathbf{x}e_n + e_n\mathbf{x} - e_n\mathbf{x}e_n)q_n = \bar{y}_n q_n, \quad x_n q_n = y_n q_n.$$

In analogy, $q_n x_n = q_n y_n$. ■

Lemma 6.4. *Let $\mathbf{u} = [u_n, e_n]$ be a unitary element of B . Then for any $k \in \mathbf{N}$ one can write $\mathbf{u}^k = [x_n, e_n]$ for a suitable sequence $\{x_n\}$.*

Proof: By Lemma 4.7, $\mathbf{u}e_n = \bar{u}_n e_n$ for all n . Let $f = \mathbf{u}e_n\mathbf{u}^*$, then $f\mathbf{u} \in \overline{T}$. By using Corollary 6.2 we obtain that $\mathbf{u}f \in \overline{T}$. Therefore

$\mathbf{u}^2 e_n = \mathbf{u} f \mathbf{u} = \mathbf{u} f f \mathbf{u} \in \overline{T}$. Now let $g = \mathbf{u} f \mathbf{u}^*$. Obviously $g \mathbf{u} \in \overline{T}$. By Corollary 6.2 it follows $\mathbf{u} g \in \overline{T}$. Hence

$$\mathbf{u}^3 e_n = \mathbf{u} \mathbf{u}^2 e_n = \mathbf{u} \mathbf{u} f \mathbf{u} = \mathbf{u} g \mathbf{u} f \mathbf{u} \in \overline{T}.$$

Inductively, applying the same k times, we obtain that $\mathbf{u}^k e_n \in \overline{T}$ and so (Corollary 6.2) $e_n \mathbf{u}^k \in \overline{T}$ for all n .

Now we can get the sequence $\{x_n\}$. As it was shown above,

$$\mathbf{u}^k e_n + e_n \mathbf{u}^k - e_n \mathbf{u}^k e_n \in \overline{T}.$$

Put $\overline{x}_n = \mathbf{u}^k e_n + e_n \mathbf{u}^k - e_n \mathbf{u}^k e_n$, where $x_n \in T$. By Proposition 6.3, $[x_n, e_n] = \mathbf{u}^k$. ■

Lemma 6.5. *Let $\{\mathbf{x}^{(k)}\}$ be a $\|\cdot\|$ -fundamental sequence in B . And let a SDD (e_n) such that $\mathbf{x}^{(k)} e_n, e_n \mathbf{x}^{(k)} \in \overline{T}$ for all n and k . Then the sequence $\{\mathbf{x}^{(k)}\}$ converges to some element $\mathbf{x} \in B$ in the norm $\|\cdot\|_1$.*

Proof: Let $\|\mathbf{x}^{(k)} - \mathbf{x}^{(l)}\|_1 \leq \varepsilon/3$. By Proposition 6.3, $\mathbf{x}^{(k)} = [y_n^{(k)}, e_n]$, where $\overline{y}_n^{(k)} = \mathbf{x}^{(k)} e_n + e_n \mathbf{x}^{(k)} - e_n \mathbf{x}^{(k)} e_n$. For fixed n , we have

$$\begin{aligned} \|y_n^{(k)} - y_n^{(l)}\| &= \|\overline{y}_n^{(k)} - \overline{y}_n^{(l)}\|_1 \\ &= \|(\mathbf{x}^{(k)} - \mathbf{x}^{(l)})e_n + e_n(\mathbf{x}^{(k)} - \mathbf{x}^{(l)}) - e_n(\mathbf{x}^{(l)} - \mathbf{x}^{(k)})e_n\|_1 \leq \varepsilon. \end{aligned}$$

Thus, $\{y_n^{(k)}\}_k$ is fundamental in T . Set $y_n = \lim_{k \rightarrow \infty} y_n^{(k)}$. Now we show that $[y_n, e_n]$ is a MO. Let $m \leq n$, then

$$\|y_n e_m - y_m e_m\| = \|(y_n - y_n^{(k)})e_m + (y_n^{(k)} - y_m^{(k)})e_m + (y_m^{(k)} - y_m)e_m\| \leq \delta(k).$$

Since $\|(y_n e_m - y_m e_m)\|$ does not depend on k we can conclude $\|y_n e_m - y_m e_m\| = 0$, $y_n e_m = y_m e_m$. In just the same way, $e_m y_n = e_m y_m$. Put $\mathbf{x} = [y_n, e_n]$. It remains to prove that the sequence $\mathbf{x}^{(k)}$ converges to \mathbf{x} in the norm $\|\cdot\|_1$. We have

$$\|\mathbf{x} - \mathbf{x}^{(k)}\|_1 = \|[y_n - y_n^{(k)}, e_n]\|_1 = \|[y_n e_n - y_n^{(k)} e_n, p_n]\|_1$$

for a suitable SDD (p_n) . Note that $\overline{y}_n^{(k)} e_n = \mathbf{x}^{(k)} e_n$. Identifying $\overline{y}_n^{(k)}$, e_n , \overline{y}_n and $\mathbf{x}^{(k)}$ with their images $\tilde{y}_n^{(k)}$, \tilde{e}_n , \tilde{y}_n and $\tilde{\mathbf{x}}^{(k)}$ in \tilde{B} , we obtain the following relations:

$$\tilde{y}_n^{(k)} \tilde{e}_n = \tilde{\mathbf{x}}^{(k)} \tilde{e}_n, \quad \tilde{y}_n \tilde{e}_n = \|\cdot\|_1 - \lim_{k \rightarrow \infty} \tilde{y}_n^{(k)} \tilde{e}_n.$$

Since $\{\tilde{\mathbf{x}}^{(k)}\}$ is a $\|\cdot\|_1$ -fundamental, there exists $\tilde{y} \in \tilde{B}$ such that $\tilde{y} = \|\cdot\|_1 - \lim_{k \rightarrow \infty} \tilde{y}^{(k)}$. Hence,

$$\tilde{y}_n \tilde{e}_n = \|\cdot\|_1 - \lim_{k \rightarrow \infty} \tilde{y}_n^{(k)} \tilde{e}_n = \|\cdot\|_1 - \lim_{k \rightarrow \infty} \tilde{\mathbf{x}}^{(k)} \tilde{e}_n = \tilde{y} \tilde{e}_n.$$

It yields

$$\begin{aligned} \|\mathbf{x} - \mathbf{x}^{(k)}\|_1 &\leq \sup_n \|y_n e_n - y_n^{(k)} e_n\| \\ &= \sup_n \|\tilde{y}_n \tilde{e}_n - \tilde{y}_n^{(k)} \tilde{e}_n\|_1 = \sup_n \|\tilde{y} \tilde{e}_n - \tilde{\mathbf{x}}^{(k)} \tilde{e}_n\| \\ &\leq \sup_n \{\|\tilde{y} - \tilde{\mathbf{x}}^{(k)}\|_1 \|e_n\|\} = \|\tilde{y} - \tilde{\mathbf{x}}^{(k)}\|_1 \rightarrow 0 \end{aligned}$$

for $k \rightarrow \infty$. ■

Theorem 6.6. *If $\mathbf{x} = \mathbf{x}^* \in B$, then there exists a positive element $\mathbf{a} \in B$ such that $\mathbf{a}^2 = \mathbf{x}^2$.*

Proof: We have $\mathbf{x} = i(1 + \mathbf{u})(1 - \mathbf{u})^{-1}$, where $\mathbf{u} = [u_n, e_n]$ is the Cayley transform of \mathbf{x} . Then $\mathbf{x}^2 = \mathbf{x}\mathbf{x}^* = (2 + \mathbf{u} + \mathbf{u}^*)(2 - \mathbf{u} - \mathbf{u}^*)^{-1}$. Observe the sequence

$$\mathbf{y}^{(l)} = \|2 + \mathbf{u} + \mathbf{u}^*\|_1^{\frac{1}{2}} \left(1 + \sum_{k=1}^l c_k (1 - (2 + \mathbf{u} + \mathbf{u}^*)/\|2 + \mathbf{u} + \mathbf{u}^*\|_1)^k \right),$$

where c_k are coefficients of Taylor series for a function $f(a) = \sqrt{1 - a}$ on $[0, 1]$. Since $(2 + \mathbf{u} + \mathbf{u}^*) \geq 0$ the sequence $\{\tilde{\mathbf{y}}^{(l)}\}$ is $\|\cdot\|_1$ -fundamental in \tilde{B} and therefore so is $\{\mathbf{y}^{(l)}\}$ in B . But all the members of the sum

$$\sum_{k=1}^l c_k (1 - (2 + \mathbf{u} + \mathbf{u}^*)/\|2 + \mathbf{u} + \mathbf{u}^*\|_1)^k$$

are linear combinations of the degrees of \mathbf{u} , \mathbf{u}^* and 1. By combining Lemma 6.4 and Lemma 6.5 the sequence $\{\mathbf{y}^{(l)}\}$ $\|\cdot\|_1$ -converges to some element $\mathbf{y} \in B$. Clearly that $\tilde{\mathbf{y}}^2 = (2 + \mathbf{u} + \mathbf{u}^*)$, hence $\mathbf{y}^2 = 2 + \mathbf{u} + \mathbf{u}^*$.

Similarly, we can find an element $\mathbf{z} \in B$ such that $\mathbf{z}^2 = 2 - \mathbf{u} - \mathbf{u}^*$. Note that all elements \mathbf{y} , $(2 - \mathbf{u} - \mathbf{u}^*)$, \mathbf{z} , $(2 + \mathbf{u} + \mathbf{u}^*)$, $(2 - \mathbf{u} - \mathbf{u}^*)^{-1}$ mutually commute. Consequently, $\mathbf{z}\mathbf{z}(2 - \mathbf{u} - \mathbf{u}^*)^{-1} = \mathbf{z}(2 - \mathbf{u} - \mathbf{u}^*)^{-1}\mathbf{z} = 1$, i.e. \mathbf{z} is invertible. Finally, putting $\mathbf{a} = \mathbf{y}\mathbf{z}^{-1}$, we obtain $\mathbf{a}^2 = \mathbf{x}^2$. Evidently, that $\mathbf{a} \in B$ and such \mathbf{a} is positive and unique. ■

7. Polar decomposition.

In this section we prove the main result of the paper: all Rickart C^* -algebras satisfy polar decomposition.

Theorem 7.1. *Let T be a finite Rickart C^* -algebra. Then the algebras B and \overline{T} coincide.*

Proof: We shall prove this statement as a spectral theorem for self-adjoint element of B . Each operator $\mathbf{x} = \mathbf{x}^* \in B$ will be approximated (in norm $\|\cdot\|_1$) by means of simple operators of \overline{T} .

For self-adjoint $\mathbf{x} \in B$ write $|\mathbf{x}| = (\mathbf{x}^2)^{\frac{1}{2}}$, $\mathbf{x}_+ = (|\mathbf{x}| + \mathbf{x})/2$, $\mathbf{x}_- = (|\mathbf{x}| - \mathbf{x})/2$. Note that $\mathbf{x}_+ - \mathbf{x}_- = \mathbf{x}$, $\mathbf{x}_+ + \mathbf{x}_- = |\mathbf{x}|$, $\mathbf{x}_+\mathbf{x}_- = 0$. If $\mathbf{x} = \mathbf{x}^* \in B$ then $\{\mathbf{x}\}_B'' = A$ is a commutative Rickart $*$ -algebra (see [4, p. 17] with C^* -norm $\|\cdot\|_1$). It is easy to see that $|\mathbf{x}|, \mathbf{x}_+, \mathbf{x}_- \in A$. ■

Lemma 7.2. *Let $\mathbf{x} \in B$, $\mathbf{x} = \mathbf{x}^*$. The family of the projections $e_\lambda = s[(\lambda 1 - \mathbf{x})_+]$ holds the following properties:*

- (a) $e_\mu \geq e_\lambda$ for $\mu \geq \lambda$;
- (b) $\sup_\lambda e_\lambda = 1$;
- (c) $\inf_\lambda e_\lambda = 0$;
- (d) If $\mu_1 \geq \mu_2 \geq \lambda_1 \geq \lambda_2$ then $(e_{\mu_1} - e_{\mu_2})(e_{\lambda_1} - e_{\lambda_2}) = 0$.

Proof: (a) Let $\lambda \leq \mu$, then $\lambda 1 - \mathbf{x} \leq \mu 1 - \mathbf{x}$. Set $\{\mathbf{x}\}_B'' = A$, $\mathbf{a} = (\lambda 1 - \mathbf{x})_+$, $\mathbf{b} = (\mu 1 - \mathbf{x})_+$. Then $\mathbf{a}, \mathbf{b} \in A$. Put $s(\mathbf{a}) = e$, $s(\mathbf{b}) = f$. By [4, p. 17], $e, f \in A$. Since \tilde{A} is commutative C^* -algebra we have $\mathbf{a} \leq \mathbf{b}$. Observe,

$$\mathbf{a}(1 - f) = (1 - f)\mathbf{a}(1 - f) \leq (1 - f)\mathbf{b}(1 - f) = 0,$$

hence $f \geq e$.

(b) Let $e = \sup_\lambda e_\lambda$. Then $\lambda(1 - e) \leq \lambda(1 - e_\lambda)$ for all $\lambda \in \mathbf{R}_+$. Further,

$$\lambda 1 - \mathbf{x} = (\lambda 1 - \mathbf{x})_+ - (\lambda 1 - \mathbf{x})_- \leq (\lambda 1 - \mathbf{x})_+.$$

In addition,

$$e_\lambda(\lambda 1 - \mathbf{x}) = e_\lambda[(\lambda 1 - \mathbf{x})_+ - (\lambda 1 - \mathbf{x})_-] = (\lambda 1 - \mathbf{x})_+.$$

Hence $\lambda(1 - e_\lambda) \leq (1 - e_\lambda)\mathbf{x}$ and thus $\lambda(1 - e) \leq (1 - e_\lambda)\mathbf{x}$.

Note since

$$(1 - e_\lambda)\mathbf{x} \leq \|x\|1,$$

consequently, $1 - e \leq \frac{\|x\|}{\lambda}$. If $e \neq 1$ then

$$1 = \|1 - e\| \leq \|x\|_1 / \lambda$$

for all $\lambda > 0$, a contradiction.

(c) Using the inequality $|\lambda|e_\lambda \leq e_\lambda x$, repeat the proof of (b).

(d) It follows immediatly from (a). ■

Now one can begin to approximate an operator x .

By Lemma $\sigma(x) \subseteq [-\|x\|_1, \|x\|_1]$. Let $\alpha \in \mathbf{R}$, $\|x\|_1 \leq \alpha$. Take an arbitrary partition of the segment $[-\alpha, \alpha]$:

$$-\alpha = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{k-1} \leq \lambda_k = \alpha.$$

Consider the elements $u_n = \lambda_n(e_{\lambda_n} - e_{\lambda_{n-1}})$. Observe that

$$\lambda(e_\mu - e_\lambda) \leq (e_\mu - e_\lambda)x \leq \mu(e_\mu - e_\lambda)$$

for $\mu \geq \lambda$. It follows that

$$\bar{u}_n - x(e_{\lambda_n} - e_{\lambda_{n-1}}) \leq \delta(e_{\lambda_n} - e_{\lambda_{n-1}}),$$

where $\delta = \max_k \{\lambda_i - \lambda_{i-1}\}$. Note $\bar{u}_n - x(e_{\lambda_n} - e_{\lambda_{n-1}}) \geq 0$. Construct an integral sum

$$\sigma = \sum_{n=1}^k \lambda_n(e_{\lambda_n} - e_{\lambda_{n-1}}).$$

Set $\lambda \geq \alpha$, then $\lambda 1 - x \geq \varepsilon 1$ for some $\varepsilon \geq 0$. Consequently, $(\lambda 1 - x)_+ = \lambda 1 - x$. Since $\lambda \notin \sigma(x)$, we obtain $s((\lambda 1 - x)_+) = 1$. So $e_\lambda = 1$ for $\lambda \geq \alpha$. In analogy, $e_\lambda = 0$ for $\lambda \leq -\alpha$. We have,

$$\bar{\sigma} - x = \sum_{n=1}^k (u_n - x(e_{\lambda_n} - e_{\lambda_{n-1}})) \leq \sum_{n=1}^k \delta(e_{\lambda_n} - e_{\lambda_{n-1}}) = \delta 1.$$

Therefore, $0 \leq \bar{\sigma} - x \leq \delta 1$, so $\|\bar{\sigma} - x\|_1 \leq \delta$. Thus, each self-adjoint operator $x \in B$ can be approximated by the simple elements from \bar{T} in the norm $\|\cdot\|_1$. It follows that \bar{T} is dense in B and therefore these C^* -algebras coincide.

Corollary 7.3. *The partial isometries are \aleph_0 -addable.*

Proof: Let (e_i) and (f_i) are sequences of ortogonal projections such that $e_i = u_i u_i^*$ and $f_i = u_i^* u_i$. Put

$$v_n = \sum_{i=1}^n u_i, k_n = \sum_{i=1}^n e_i, t_n = \sum_{i=1}^n f_i, e = \bigvee_i e_i, f = \bigvee_i f_i.$$

Then the sequences $(p_n = e^\perp + k_n)$ and $(q_n = f^\perp + t_n)$ are SDD. Set $d_n = p_n \wedge q_n$. Clearly that $\mathbf{v} = [v_n, d_n]$ is MO from B . By previous theorem, there exists $v \in T$ such that $\bar{v} = \mathbf{v}$. It is easy to see that $v u_i^* u_i = u_i$, $u_i u_i^* v = u_i$ for all $i \in \mathbf{N}$. ■

Corollary 7.4. *All Rickart C^* -algebras satisfy polar decomposition.*

Proof: By [1, Th. 3.4], this assertion is reduced to a finite case. Now combine Corollary 7.3 and [1, Prop. 2.1] and the Corollary follows. ■

Corollary 7.5. *Let T be a Rickart C^* -algebra, then the matrix algebras $M_n(T)$ over T are also Rickart C^* -algebras for all $n \in \mathbb{N}$.*

Proof: See [1, Th. 3.5]. ■

8. Axiom (PSR) in Q .

Using Theorem 7.1 and the methods of [3], [11] or [6], we can describe the self-adjoint elements in Q .

Theorem 8.1. *Let $\mathbf{x} = \mathbf{x}^* \in Q$, $\mathbf{u} = \bar{u}$ ($u \in T$) its Cayley transform. One can write $\mathbf{x} = [x_n, e_n]$ with $x_n, e_n \in \{u\}''$, $x_n^* = x_n$, $x_n e_n = x_n$, $x_n^2 \uparrow$.*

Proof: See [3, Th. 4.2]. ■

An element $\mathbf{x} \in Q$ is positive, written $\mathbf{x} \geq 0$, if $\mathbf{x} = \mathbf{y}^* \mathbf{y}$ for some $\mathbf{y} \in Q$.

Theorem 8.2. *Let $\mathbf{x} = \mathbf{x}^* \in B$, $\mathbf{u} = \bar{u}$ its Cayley transform. The following conditions are equivalent:*

- a) $\mathbf{x} \geq 0$;
- b) one can write $\mathbf{x} = [y_n, f_n]$ with $y_n \geq 0$;
- c) the spectrum of u contained in $\{e^{i\Theta} : -\pi \leq \Theta \leq 0\}$;
- d) one can write $\mathbf{x} = [x_n, e_n]$ with $x_n, e_n \in \{u\}''$, $x_n \geq 0$, $x_n e_n = x_n$.

Proof: See [3, Th. 6.1]. ■

Corollary 8.2. *Q satisfies axiom (PSR).*

Proof: See [3, Cor. 6.2]. ■

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